JOURNAL OF COMPUTATIONAL PHYSICS 29, 133–140 (1978)

Note

Local Mesh Refinement with Finite Elements for Elliptic Problems

1. Introduction

This paper is concerned with finite element methods based on successive mesh refinement for the effective treatment of singularities in two dimensional second and fourth order elliptic boundary value problems. In each case the problem is defined in a rectangular domain Ω, with boundary $\partial\Omega$, and the singularity arises on account of $\partial\Omega$ possessing a *re-entrant* corner. Both problems are restated in weak form in a suitable Sobolev space setting ($W_2^m(\Omega)$; $m = 1, 2$ respectively for the second and fourth order problems). The domain Ω is partitioned into non-overlapping elements and with a suitable finite dimensional space S^h the Galerkin technique is applied to produce an approximation $u_h \in S^h$ to the weak solution $u \in W_2^m(\Omega)$, where $S^h \subset W_2^m(\Omega)$ because only conforming trial functions are used.

The presence of the singularity causes the finite element approximation u_h to be inaccurate in the neighbourhood of the re-entrant corner. Many adaptations of the standard method have been suggested for improving this accuracy; for example the use of singular functions to augment the trial function space, see, e.g., Fix [4], Barnhill and Whiteman [2], and Griffiths [6], and the use of special isoparametric elements at the corner, see e.g. Akin [1], Henshell and Shaw [7] and Wait [8]. We propose here to consider the method of *successive local mesh refinement* in the neighborhood of the singularity.

Finite element trial functions are usually derived from piecewise polynomial interpolants. Thus in Section 2 we consider bilinear-C^0 and bicubic-C^1 interpolants on the standard square ($[0, 1] \times [0, 1]$) and show how these can be adapted to produce 5-node-C^0 and 20-node-C^1 elements. Examples of the application of refinement schemes based on these interpolants for harmonic and biharmonic problems respectively in regions containing slits are given in Section 3.

2. Conforming C^0 and C^1 Trial Functions; Mesh Refinement

A key step in the successful application of the Galerkin method is the construction of the trial function space S^h. This generally consists of functions which are polynomial in each element of the partition of Ω and satisfy over Ω some global continuity property—the conforming condition; in the current cases for example this condition is that u_h be in $C^0(\bar{\Omega})$ for the second order problem and that u_h be in $C^1(\bar{\Omega})$ for the

0021-9991/78/0291–0133$02.00/0

fourth order problem, where $\bar{\Omega}$ is the closure of Ω. These conditions ensure that in each case S^h is a subspace of the appropriate Sobolev space.

In each element e of the partition the approximating function $u_h(x, y)|_e$ is derived from an interpolating polynomial $P[u(x, y)]$ which interpolates the values of u (Lagrange case), and frequently also certain derivatives of u (Hermite case), at nodes in the element. In order that u_h may satisfy the conforming condition over Ω, a suitable choice of nodes and interpolation conditions for P in each element has to be made. The normal process is to treat each element in the physical plane separately by mapping it onto a *standard element* $E \equiv [0, 1] \times [0, 1]$ in the ξ, η plane with an affine transformation. We therefore now consider interpolation on this standard element.

Let the univariate Nth order Hermite interpolation operator P^t be defined as

$$P^t[f(t)] = \sum_{i=0}^{N} \phi_i(t) f^{(i)}(0) + \sum_{i=0}^{N} \psi_i(t) f^{(i)}(1), \tag{1}$$

where the $\{\phi_i(t)\}_{i=0}^N$ and $\{\psi_i(t)\}_{i=0}^N$ are the cardinal basis functions of the Hermite two point interpolation on [0, 1] and the $f^{(i)}(z)$ are point evaluations of the ith order derivatives of f at the point z. In the square E a bivariate Nth order Hermite interpolation operator can be found by taking the tensor product $P^\xi P^\eta$ so as to form the interpolating polynomial

$$\begin{aligned} p(\xi, \eta) &\equiv P[F(\xi, \eta)] \equiv P^\xi P^\eta[F(\xi, \eta)] \\ &= \sum_{i,j \leqslant N} \phi_i(\xi)\, \phi_j(\eta)\, F_{i,j}(0, 0) + \sum_{i,j \leqslant N} \phi_i(\xi)\, \psi_j(\eta)\, F_{i,j}(0, 1) \\ &\quad + \sum_{i,j \leqslant N} \psi_i(\xi)\, \phi_j(\eta)\, F_{i,j}(1, 0) + \sum_{i,j \leqslant N} \psi_i(\xi)\, \psi_j(\eta)\, F_{i,j}(1, 1), \end{aligned} \tag{2}$$

where the interpolation space is the set of monomials

$$\{\xi^i \eta^j;\ 0 \leqslant i, j \leqslant 2N + 1\},$$

and

$$F_{i,j} \equiv \frac{\partial^{i+j} F}{\partial \xi^i\, \partial \eta^j}.$$

The case $N = 0$ is that of bilinear interpolation with

$$\phi_0(t) = (1 - t) \qquad \text{and} \qquad \psi_0(t) = t. \tag{3}$$

The case $N = 1$ is that of bicubic Hermite interpolation with cardinal basis functions

$$\begin{aligned} \phi_0(t) &= (t - 1)^2(2t + 1), \\ \phi_1(t) &= (t - 1)^2\, t, \\ \psi_0(t) &= \phi_0(1 - t) = t^2(-2t + 3), \\ \psi_1(t) &= -\phi_1(1 - t) = t^2(t - 1). \end{aligned} \tag{4}$$

The use of the interpolant (2) in every element of a *regular* partition of Ω, with in each case a suitable transformation of co-ordinates, produces over Ω a piecewise polynomial function with C^N continuity.

However, the partition may not be regular, as for example when the mesh is refined locally in some subdomain of Ω by (say) a successive halving procedure as in Fig. 1. In this case interpolant (2) cannot be used as it stands in every element, because some elements of the mesh possess *hanging points*, such as the point $(\frac{1}{2}, 0)$ illustrated in the standard element E of Fig. 2. We wish on this nonregular mesh to produce a function which is $C^N(\Omega)$. Our method, which is described below, is not the only

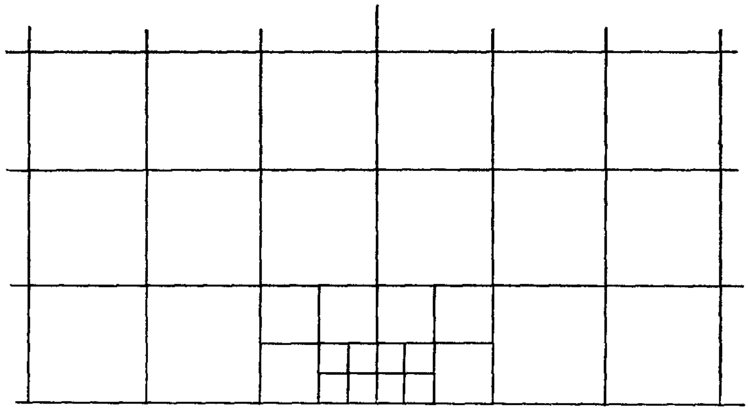

FIG. 1. Local Mesh refinement.

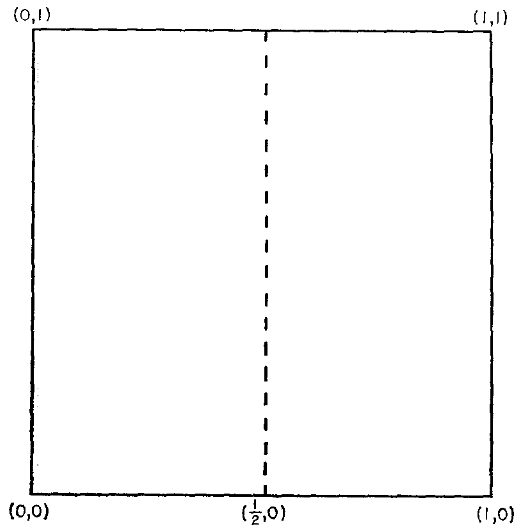

FIG. 2. Five node "standard" element.

technique for doing this. However, its merits, when compared with an obvious alternative, are discussed in Section 4.

In order to produce our function which is $C^N(\Omega)$, we have to construct an interpolant which maintains C^N continuity over the "five-node standard element" E of Fig. 2 and its edges. We achieve this by introducing into this element an imaginary node at $(\frac{1}{2}, 1)$. The two halves of the element $(\xi \gtrless \frac{1}{2})$ are then considered separately. For each, consideration of the continuity across $\eta = 1$ demands that at the point $(\frac{1}{2}, 1)$

$$p_{i,j}(\tfrac{1}{2}, 1) \equiv p_{i,j} = \sum_{k=0}^{N} \phi_k^{(i)}(\tfrac{1}{2})\, F_{k,j}(0, 1) + \sum_{k=0}^{N} \psi_k^{(i)}(\tfrac{1}{2})\, F_{k,j}(1, 1), \qquad 0 \leqslant i, j \leqslant N. \tag{5}$$

This value can then be inserted into the respective local interpolation functions over $[0, \frac{1}{2}] \times [0, 1]$ and $[\frac{1}{2}, 1] \times [0, 1]$, where these are given by (2) after suitable changes of scale and origin. For example on $[0, \frac{1}{2}] \times [0, 1]$, let

$$\begin{aligned} p(\xi, \eta) = &\sum_{i,j \leqslant N} \phi_i(2\xi)\, \psi_j(\eta)\, 2^{-i} F_{i,j}(0, 0) \\ &+ \sum_{i,j \leqslant N} \psi_i(2\xi)\, \phi_j(\eta)\, 2^{-i} F_{i,j}(\tfrac{1}{2}, 0) \\ &+ \sum_{i,j \leqslant N} \phi_i(2\xi)\, \psi_j(\eta)\, 2^{-i} F_{i,j}(0, 1) \\ &+ \sum_{i,j \leqslant N} \psi_i(2\xi)\, \psi_j(\eta)\, 2^{-i} p_{i,j}\,, \end{aligned} \tag{6}$$

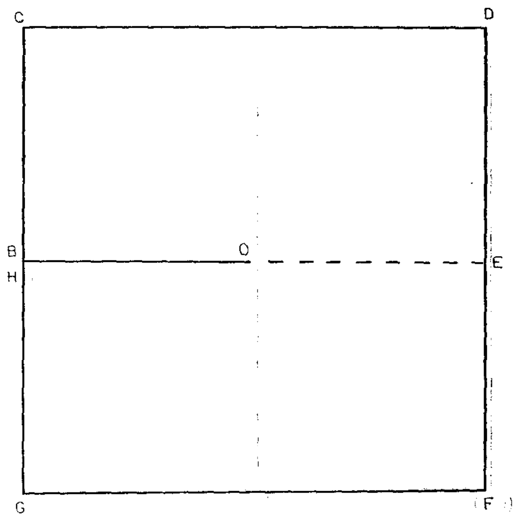

FIGURE 3

where in (6) the $p_{i,j}$ is defined by (5). For the cases $N = 0$ and 1, the ϕ's and ψ's are as in (3) and (4), respectively. The combination of (6) in $[0, \frac{1}{2}] \times [0, 1]$ together with its dual in $[\frac{1}{2}, 1] \times [0, 1]$ produces a five-node interpolant in E with the required continuity.

On the mesh with local refinement as in Fig. 1, suitable transformations of the above five node element as required, and of interpolant (2) for every four node element, produce the desired piecewise polynomial $C^N(\Omega)$ function. The trial functions of the Galerkin method on the refined meshes are chosen from the spaces of interpolating functions of this type. Two examples of the use of the refinement scheme are now given.

3. Applications of the Refinement Scheme

3.1 *Harmonic Problem*

The function $u(x, y)$ satisfies

$$
\begin{aligned}
-\Delta[u(x, y)] &= 0, && (x, y) \in \Omega, \\
u(x, y) &= 500, && (x, y) \in \overline{BC}, \\
u(x, y) &= 0, && (x, y) \in \overline{EO}, \\
\frac{\partial u(x, y)}{\partial y} &= 0, && (x, y) \in \overline{OB} \cup \overline{CD}, \\
\frac{\partial u(x, y)}{\partial x} &= 0, && (x, y) \in \overline{DE},
\end{aligned}
\tag{7}
$$

where Ω is the rectangular region OBCDEO of Fig. 3, which for this problem is such that $EO = OB = BC = \frac{1}{2}$.

This problem is derived using symmetry from a well known and much studied harmonic problem in the whole rectangle OBCDEFGHO containing a slit, and which contains a singularity at 0 (see [9–13]).

The Galerkin method is used to solve the weak form of (7). As this method is now so well known, having appeared in text books and in many survey articles, it will not be described in detail here. The Galerkin method is applied using over Ω a square mesh of side $h = 1/14$ and with C^0 trial functions based on (2) and (3). Numerical results so obtained are given at the three specific points $P = (0, 1/14)$, $Q = (-1/14, 0)$ and $R = (\frac{3}{7}, \frac{3}{7})$ of Ω in Fig. 4. These should be compared with accurate results obtained by Whiteman and Papamichael [13] using a conformal transformation method (CTM), which are also given in Fig. 4. It will be seen that with no refinement accuracy is poor, especially in the neighborhood of 0. In order to improve this we use local mesh refinement based on the *refinement scheme* of Section 2.

The conforming condition for harmonic problems demands that trial functions be in $C^0(\Omega)$. Thus the refinement scheme is applied with $N = 0$ in (2) and (6). Levels of

Number of Levels of Refinement	Value of $U(x, y)$ at		
	$P \equiv (0, \frac{1}{14})$	$Q \equiv (-\frac{1}{14}, 0)$	$R \equiv (\frac{3}{7}, \frac{3}{7})$
0 ($h = \frac{1}{14}$)	97.05	147.05	88.73
1	99.61	150,52	89.78
2	101.62	153.39	90.31
3	102.72	154.92	90.57
4	103.27	155.69	90.70
5	103.54	156.07	90.78
6	103.68	156.26	90.80
7	103.75	156.36	90.82
8	103.78	156.40	90.83
CTM [13] Results	103.77	156.48	91.34

FIG. 4. Harmonic problem; each level of refinement produces eight extra mesh points.

refinement ranging from 1 to 8 are used, and the respective results, given in Fig. 4, show an improvement of accuracy as refinement increases. It is seen that the stage has been reached where the Galerkin solutions are more accurate near the singularity than they are at points remote from 0. The effect of the singularity on the numerical solution has thus been neutralized by the refinement, and the errors on the coarse mesh are due to the mesh spacing.

3.2 *Biharmonic Problem*

The function $u(x, y)$ satisfies

$$\begin{aligned}
&\Delta^2[u(x, y)] = 0, && (x, y) \in \Omega, \\
&u(x, y) = 0, \quad \frac{\partial u(x, y)}{\partial y} = 0, && (x, y) \in \overline{OB}, \\
&u(x, y) = 0, \quad \frac{\partial u(x, y)}{\partial x} = 0, && (x, y) \in \overline{BC}, \\
&u(x, y) = \sigma\left(\frac{x^2}{2} + ax + \frac{a^2}{2}\right), && \frac{\partial u}{\partial y} = 0, \quad (x, y) \in \overline{CD}, \\
&u(x, y) = 2\sigma a^2, \quad \frac{\partial u}{\partial x} = 2\sigma a, && (x, y) \in \overline{DE},
\end{aligned} \tag{8}$$

where Ω is again the slit rectangular region of Fig. 3, and the appropriate symmetric boundary values are assumed on the lower rectangle. This problem is that of a two-dimensional elastic solid in a rectangle containing a crack subjected to an inplane load σ. For the model problem in question we take BO = OE = a = .4, BC = .7 and $\sigma = 10^4$, and apply the Galerkin technique to (8) with the refinement scheme of Fig. 1 using a $C^1(\Omega)$ function, based on the interpolants (2) and (6) with $N = 1$, and with a coarse mesh of length $h = .1$. Levels of refinement ranging from 1 to 7 are used and the results, given in Fig. 5, show the improvement in accuracy. The comparison in this case is with numerical values obtained by Bernal and Whiteman [3] using an adaptation of the standard thirteen-point finite difference replacement for the biharmonic operator through the use of singular functions having the form of the dominant part of the singularity in the neighborhood of the end of the slit.

Number of Levels of Refinement	Value of $U(x, y)$ at		
	$P \equiv (0, \frac{1}{10})$	$Q \equiv (\frac{1}{10}, 0)$	$R \equiv (-\frac{3}{10}, \frac{6}{10})$
0 ($h = \frac{1}{10}$)	133.1	470.4	47.7
1	139.7	488.7	47.7
2	143.1	498.5	47.8
3	144.8	503.3	47.8
4	145.7	505.7	47.8
5	146.1	506.9	47.8
6	146.3	507.5	47.8
7	146.4	507.8	47.8
F-D [3] Results	147	508	48

FIG. 5. Biharmonic problem; each level of refinement introduces seventeen extra mesh points as symmetry has not been exploited.

4. Discussion

The refinement method proposed here has to be compared with the other finite element techniques listed in Section 1. It has the advantage that it is simple to implement, and in particular it is much simpler to program than the method of augmenting the trial function space with singular functions. The basic simplicity of this method stems from the repetitive nature of the refinement. The convergence of refined results for the problem defined by (8) to those of [3] is significant, considering that the methods are in no way related.

An important though simple point is that in our five node element, Fig. 2, the values at the node $(\frac{1}{2}, 0)$ have been introduced as unknowns. A more obvious technique would be to replace the unknown values at $(\frac{1}{2}, 0)$ by the Hermite interpolants between values at $(0, 0)$ and $(1, 0)$. This would have the effect of introducing the coarse mesh spacing into the refined mesh. Thus the benefit of the higher local accuracy of the refined mesh is not gained immediately, since the domain of influence of the coarse mesh spreads into that of the fine mesh. Numerical experiments by Galliara [5] with the problem defined by (7) verify this fact.

From the results of Section 3 we feel that this refinement is a viable alternative to other techniques for dealing with boundary singularities.

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Received: May 23, 1974; revised November 2, 1977

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